3101. (a) Since $x = k$ is a line of symmetry, the integral from 0 to k is equal to the integral from k to $2k$. Hence, $I_1 = 2$.

(b) Applying a stretch by scale factor 2 in the x direction, the original integral is transformed into this one. The area is scaled by the same factor, so $I_2 = 2$.

3102. By the chain rule,

$$\text{f}'(x) = -xe^{-\frac{1}{2}x^2}.$$

By the product and chain rules,

$$\text{f}''(x) = (x^2 - 1)e^{-\frac{1}{2}x^2}.$$

This is zero, and changes sign, at $x = \pm 1$, giving points of inflection at $\left(\pm 1, e^{-\frac{1}{2}}\right)$. The gradients at these points are $m = \mp e^{-\frac{1}{2}}$. The line $y\sqrt{e} = 2 - x$ goes through $\left(1, e^{-\frac{1}{2}}\right)$, with gradient $m = e^{-\frac{1}{2}}$, so, as required, it is tangent to the curve at one of its points of inflection.

3103. The cube has $V = 6$, $E = 12$, $F = 8$, which gives an Euler characteristic of $\chi = 6 - 12 + 8 = 2$. The tetrahedron has $V = 4$, $E = 6$, $F = 4$, which gives an Euler characteristic of $\chi = 4 - 6 + 4 = 2$, as required.

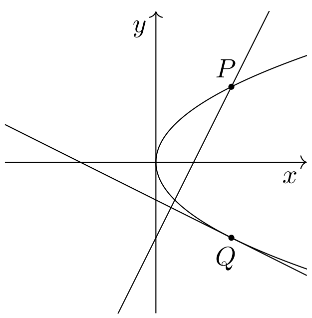
— Nota Bene —

The Euler characteristic of *every* polyhedron is in fact the same. This is a classic result in *topology*: the study of fundamental structure.

3104. (a) The probability of drawing two white, two black is ${}^4\text{C}_2 \times \frac{4}{8} \times \frac{3}{7} \times \frac{4}{6} \times \frac{3}{5} = \frac{18}{35}$.

(b) The probability of drawing three white, one black is ${}^4\text{C}_3 \times \frac{4}{8} \times \frac{3}{7} \times \frac{2}{6} \times \frac{4}{5} = \frac{8}{35}$. We double this, giving $\frac{16}{35}$.

3105. The scenario is



Using calculus, the tangent T at Q has equation $y = -\frac{1}{2}x - \frac{1}{2}$. So, the perpendicular to T which passes through P has equation $y = 2x - 1$. Solving these equations simultaneously, we get $(1/5, -3/5)$.

3106. We can express the transformation as a stretch in the x direction and a stretch in the y direction by writing, for positive constant a, b,

$$x^3 - 8x \equiv a\big((bk)^3 - bk\big).$$

Equating coefficients,

$$\begin{aligned} x^3 &: 1 = ab^3 \\ x^1 &: -8 = -ab. \end{aligned}$$

Dividing the equations, $b = 8^{-\frac{1}{2}}$, so $a = 8^{\frac{3}{2}}$. The area scale factor, then, is $\frac{a}{b} = 8^{\frac{3}{2}} \times 8^{\frac{1}{2}} = 64$.

3107. In the diagram, the beam could be in equilibrium with either

① a reaction force of mg at the central support and zero at the outer supports, or

② reaction forces of $\frac{1}{2}mg$ at the outer supports and zero at the central support.

Since both of these (and, in fact, infinitely many other combinations) are feasible, it is not possible to determine the reactions from the information given.

3108. We are told that the point $(q, 0)$ is a stationary point on the x axis (double root), which is a local minimum. Hence, either side of this point, $\text{f}(x)$ must be positive. And, being a negative quartic, $\text{f}(x) \to -\infty$ as $x \to \pm\infty$. Therefore, there must be two more roots either side of $(q, 0)$. This gives three distinct roots. And, since $(q, 0)$ is a double root, there can be no more.

3109. The odd-powered statements are false: $x = 3\pi/2$ provides a counterexample in both of those cases. The even-powered statement is true: since $\sin^2 x$ is non-negative, applying a mod function to it has no effect.

(a) False,

(b) True,

(c) False.

3110. Differentiating both sides,

$$\frac{2}{y} = \frac{dy}{dx}.$$

Separating the variables,

$$\begin{aligned} &\int 2\,dx = \int y\,dy \\ \implies &\; 2x + c = \tfrac{1}{2}y^2 \\ \implies &\; y = \pm\sqrt{4x + d}. \end{aligned}$$

3111. Substituting, we need the following equation to have at least one real root:

$$\begin{aligned} & x^3 + (k-x)^3 = 1 \\ \Longrightarrow\ & k^3 - 3k^2x + 3kx^2 = 1 \\ \Longrightarrow\ & 3kx^2 - 3k^2x + k^3 - 1 = 0. \end{aligned}$$

If $k = 0$, then this is $-1 = 0$, and has no roots. If $k \neq 0$, then it is a quadratic in x. We need at least one root. Setting $\Delta \geq 0$,

$$\begin{aligned} & 9k^4 - 12k(k^3 - 1) \geq 0 \\ \Longrightarrow\ & -3k^4 + 12k \geq 0. \end{aligned}$$

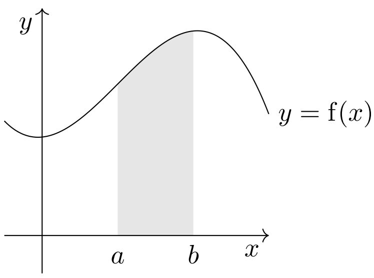
The boundary equation has roots $k = 0$ or $k = \sqrt[3]{4}$. We require a negative quartic to be positive, so, excluding zero as described earlier, $k \in \left(0, \sqrt[3]{4}\right]$.

3112. In algebra, the initial statement is $\mathrm{g}(-x) \equiv \mathrm{g}(x)$. Differentiating by the chain rule,

$$\begin{aligned} & -\mathrm{g}'(-x) \equiv \mathrm{g}'(x) \\ \Longrightarrow\ & \mathrm{g}'(-x) \equiv -\mathrm{g}'(x). \end{aligned}$$

So, $\big(-x, \mathrm{g}'(-x)\big)$ can be written as $\big(-x, -\mathrm{g}'(x)\big)$, which is the image of $\big(x, \mathrm{g}'(x)\big)$ under rotation by 180° around the origin. Hence, the graph $y = \mathrm{g}'(x)$ has rotational symmetry around the origin. □

3113. Consider the following sketch:



The integral calculates the (signed) area of the shaded region. This is the continuous sum of the value of $\mathrm{f}(x)$ across the domain $[a, b]$. To calculate the average value (average height), we divide this area by the width of the domain, which is $b - a$.

3114. Let θ be the angle of projection. The maximal height is given by

$$\begin{aligned} & 0 = u^2 \sin^2\theta - 2gh_{\max} \\ \Longrightarrow\ & h_{\max} = \frac{u^2 \sin^2\theta}{2g}. \end{aligned}$$

So, successful outcomes require

$$\begin{aligned} & \frac{u^2 \sin^2\theta}{2g} \geq \frac{3u^2}{8g} \\ \Longrightarrow\ & \sin^2\theta \geq \tfrac{3}{4}. \end{aligned}$$

For $\theta \in [0, 90°]$, this is $\theta \geq 60°$. So, the probability is $30°/90° = 1/3$.

3115. (a) The discriminant of the denominator is -3, which is negative. So, the denominator is never zero, which means the function has domain $\mathbb{R}$.

(b) The function f is defined over $\mathbb{R}$, so $y = \mathrm{f}(x)$ has no vertical asymptotes. The x axis is an asymptote, since $y \to 0$ as $x \to \pm\infty$. Hence, the range must have a bound at a stationary point. By the quotient rule,

$$\frac{dy}{dx} = \frac{-x(x+2)}{(x^2+x+1)^2}.$$

Setting the numerator to zero, there are SPs at $(0, 1)$ and $(-2, -1/3)$. These are above and below the horizontal asymptote (the x axis), so the range of f is $[-1/3, 1]$.

3116. The x intercepts show reflection in the y axis, and the vertices show reflection in the x axis.

Applying these algebraically to $y = px^2 + qx + r$, we replace x by $-x$ and y by $-y$. This gives

$$\begin{aligned} & -y = p(-x)^2 + q(-x) + r \\ \Longrightarrow\ & y = -px^2 + qx - r. \end{aligned}$$

— Nota Bene —

Reflection in the x axis, and also in the y axis, is equivalent to rotation by 180° around the origin.

3117. Since the gradient is non-zero, we can immediately rule out degree 0, which is a constant function.

Assume, for a contradiction, that f has degree 2. Then f′ has degree 1, so the equation $\mathrm{f}'(x) = 2$ is linear. But we are told that $\mathrm{f}'(x) = 2$ has two roots. This is a contradiction.

Hence, f cannot have degree 0 or 2, as required.

3118. The expression is a cubic in $(a^3 + 1)$, of the form

$$(a^3+1)^3 + b(a^3+1)^2 + c(a^3+1) + d.$$

Equating coefficients,

$$\begin{aligned} a^6 &: 3 + b = 5 \\ a^3 &: 3 + 2b + c = 7 \\ a^0 &: 1 + b + c + d = 3. \end{aligned}$$

From the top, $b = 2$, $c = 0$, $d = 0$. So,

$$a^9 + 5a^6 + 7a^3 + 3 \equiv (a^3+1)^3 + 2(a^3+1)^2.$$

— Alternative Method —

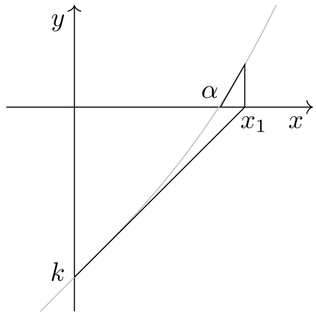
Let $x = a^3 + 1$, which gives $a^3 = x - 1$. Substituting this in,

$$\begin{aligned} & a^9 + 5a^6 + 7a^3 + 3 \\ =\ & (x-1)^3 + 5(x-1)^2 + 7(x-1) + 3 \\ \equiv\ & x^3 + 2x^2 \\ =\ & (a^3+1)^3 + 2(a^3+1)^2. \end{aligned}$$

3119. (a) The derivative is $f'(x) = 3x^2 + 1$, which is +ve. So, the function is increasing, and has exactly one root. And, since $k < 0$ (the y intercept of $y = f(x)$ is negative), we know that $\alpha > 0$ (the x intercept of $y = f(x)$ is positive).

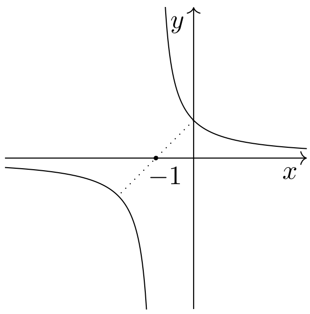
(b) The second derivative is $f''(x) = 6x$. This is zero and changes sign at $x = 0$, so $(0, k)$ is a point of inflection. And, for $x > 0$, the second derivative is positive, so the graph is convex.

(c) Sketching the first two iterations of the N-R method, we have



The graph is convex for $x > 0$, Hence, the first iteration will give $x_1 > \alpha$, as shown above. From that point, every subsequent iteration will give $x_{n+1} < x_n$, and thus the method will converge back to $x = \alpha$.

3120. The graph $y = \frac{1}{x}$ has rotational symmetry about the origin, and so the shortest path between its two sections must pass through that point. (In fact, the path is the line $y = x$). The graph $y = \frac{1}{x+1}$ is a translation of $y = \frac{1}{x}$ by vector $-\mathbf{i}$, so the shortest path must pass through $(-1, 0)$.



3121. (a) Using various trig identities,

$$\begin{aligned} &\cos 3x \\ \equiv\ &\cos(2x + x) \\ \equiv\ &\cos 2x \cos x - \sin 2x \sin x \\ \equiv\ &(2\cos^2 x - 1)\cos x - 2\sin^2 \cos x \\ \equiv\ &(2\cos^2 x - 1)\cos x - 2(1 - \cos^2 x)\cos x \\ \equiv\ &4\cos^3 x - 3\cos x. \end{aligned}$$

Substituting this in,

$$\begin{aligned} &\cos x + 3\cos 3x \\ \equiv\ &\cos x + 3(4\cos^3 x - 3\cos x) \\ \equiv\ &12\cos^3 x - 8\cos x, \text{ as required.} \end{aligned}$$

(b) The derivative is $\cos x + 3\cos 3x$. So, for SPs,

$$\begin{aligned} &12\cos^3 x - 8\cos x = 0 \\ \implies &\cos x(3\cos^2 x - 2) = 0 \\ \implies &\cos x = 0, \pm\sqrt{2/3}. \end{aligned}$$

For $x \in [0, \pi]$, this gives SPs at $(\pi/2, 0)$ and, to 3sf, $(0.615, 1.54)$ and $(2.53, 1.54)$.

3122. The polynomial f is convex, so $f''(x) > 0$ for all $x \in \mathbb{R}$. This means the polynomial equation $f''(x) = 0$ must have no real roots. An equation of odd degree must have a root, so we know that f'' has even degree. Integrating twice raises the degree by 2, so f must also have even degree. □

3123. (a) Since the string has length 6, the triangles formed at each side are equilateral. Hence, the oblique sections are at inclination 30° to the horizontal. So, $10\sqrt{3} = 2T\cos 30°$, which gives $T = 10$ N.

(b) On each peg, the string exerts two tensions, with 150° between them. Resolving along the angle bisector, the resultant force applied is $20\cos 75°$ N. Using $\cos\theta \equiv \sin(90° - \theta)$, we can rewrite this as $20\sin 15°$ N.

3124. (a) Rearranging to $x = \ln x + 2$, we set up

$$x_{n+1} = \ln x_n + 2.$$

Running this with $x_0 = 1$ gives $x_1 = 2$ and $x_2 = 2.69$. This converges: $x_n \to 3.14619...$. This is 3.146 to 4sf. Calculating error bounds, with $f(x)$ representing the LHS of the original equation,

$$\begin{aligned} f(3.1455) &= -0.00047... < 0, \\ f(3.1465) &= 0.00020... > 0. \end{aligned}$$

There is a sign change, so $x = 3.146$ (4sf).

(b) Differentiating, $g'(x) = 1 - \frac{1}{x}$. For $x \approx 0.1$, the gradient is around $g'(0.1) = -9.9$. Since $|g'(x)| > 1$, the iteration will diverge from the root at $x \approx 0.1$.

3125. By Pythagoras, the squared distance between $(\sqrt{6}, \sqrt{2})$ and each of the other two vertices is

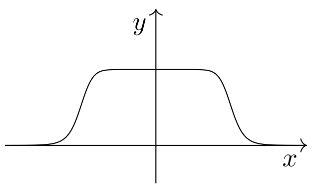
$$16 - 4\sqrt{2} - 4\sqrt{6}.$$

So, this must be the squared side length of the polygon. The squared distance between the two non-adjacent vertices is $16 - 8\sqrt{3}$. By the cosine rule, the interior angle is

$$\arccos \tfrac{2(16-4\sqrt{2}-4\sqrt{6})-(16-8\sqrt{3})}{2(16-4\sqrt{2}-4\sqrt{6})} = 165°.$$

The exterior angle is $\frac{360°}{n} = 15°$, so $n = 24$.

3126. For $x \in (-1, 1)$, x^{2k} is very small, so $y \approx 1$. For $x = \pm 1$, $y = 1/2$. For $x \notin [-1, 1]$, x^{2k} is very large, so $y \approx 0$. For large k, the behaviour therefore approaches that of a step function:



3127. Quoting $\frac{d}{dx} \operatorname{cosec} x = -\cot x \operatorname{cosec} x$, we use the reverse chain rule to get

$$\int \cot 2x \operatorname{cosec} 2x \, dx = -\tfrac{1}{2} \operatorname{cosec} 2x + c.$$

——— Nota Bene ———

Integrals using the reverse chain rule (inspections) are best understood by differentiation of the end result.

3128. Call the bottom left point $(0, 0)$, the top right point (m, n), and the other lattice point on the diagonal (a, b). Then the gradients $\frac{n}{m}$ and $\frac{b}{a}$ are equal, so $\frac{n}{m} = \frac{b}{a}$. The latter fraction is in lower terms than the former, which means m and n must have a common factor. QED.

3129. (a) To find $\bar{y}$, we need to find $\sum y_i$. This is $\sum(ax_i^2 + b)$. We can split this up as follows:

$$\begin{aligned} &\sum(ax_i^2 + b) \\ \equiv\ &\sum ax_i^2 + \sum b \\ \equiv\ &a \sum x_i^2 + bn. \end{aligned}$$

The variance formula is $s_x^2 = \frac{1}{n}\sum x_i^2 - \bar{x}^2$, which we can rearrange to $\sum x_i^2 = n(s_x^2 + \bar{x}^2)$. Substituting this in,

$$\bar{y} = n(as_x^2 + \bar{x}^2 + b).$$

(b) For the variance s_y^2, we need to calculate

$$\begin{aligned} &\sum(ax_i^2 + b)^2 \\ \equiv\ &\sum a^2 x_i^4 + ... \end{aligned}$$

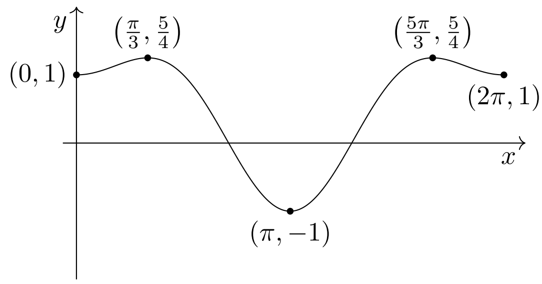
The sum of the fourth powers of x_i is info that is not summarised in the mean $\bar{x}$ and standard deviation s_x. Hence, it is impossible to find an expression for s_y in terms of the information given.

3130. The first derivative is $\sin x(2\cos x - 1)$, so the curve has SPs where $\sin x = 0$ and where $\cos x = \frac{1}{2}$. For $x \in [0, 2\pi)$, this is

$$x = 0, \tfrac{\pi}{3}, \pi, \tfrac{5\pi}{3}, 2\pi.$$

The second derivative is $-\cos x + 2\cos 2x$. At the SPs, this takes the values $1, -3/2, 3, -3/2, 1$.

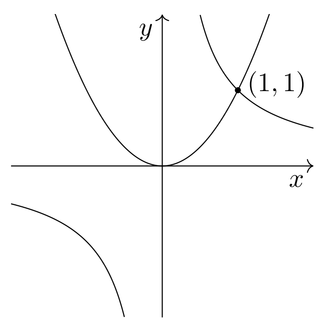
So, there are maxima at $(\pi/3, 5/4)$ and $(5\pi/3, 5/4)$, and minima at $(0, 1)$, $(\pi, -1)$ and $(2\pi, 1)$. Hence, the graph is



3131. Setting up the boundary equation,

$$\begin{aligned} & x^2 - \frac{1}{x} = 0 \\ \Longrightarrow\ & x^3 - 1 = 0 \\ \Longrightarrow\ & x = 1. \end{aligned}$$

Rearranging to $x^2 \geq \frac{1}{x}$, we sketch each side:



We need the x values for which the parabola is above (or on) the hyperbola. So, the solution set is $(-\infty, 0) \cup [1, \infty)$.

3132. (a) A counterexample is $\mathrm{f}(x) = x$ and $\mathrm{g}(x) = -x$. These give $R = \{0\}$ and $S = \{0\}$. The union of these is not $\mathbb{R}$.

(b) A counterexample is both functions defined as the zero function. Both S and R are then $\mathbb{R}$, as is their intersection.

3133. (a) At (p, p^2), the gradient of $y = x^2$ is $2p$, so the gradient of the normal is $-\frac{1}{2p}$. The formula $y - y_1 = m(x - x_1)$ gives the equation of the normal as $y = -\frac{1}{2p}x + p^2 + \frac{1}{2}$.

(b) The shortest distance to the parabola is on the normal.

So, we first substitute in $(0, 2)$, which gives $2 = p^2 + \frac{1}{2}$, so $p = \pm\sqrt{1.5}$. The (equal) closest point to $(0, 2)$ is $(\sqrt{1.5}, 1.5)$. By Pythagoras, the distance is 1.33 (3sf).

Next, we substitute in $(2, -0.5)$. This gives $-0.5 = -\frac{1}{p} + p^2 + \frac{1}{2}$, which is $p^3 + p - 1 = 0$, a cubic in p. Using a calculator, $p = 0.682$ (3sf). By Pythagoras, the distance between $(0.682, 0.682^2)$ and $(2, -0.5)$ is $1.63 > 1.33$.

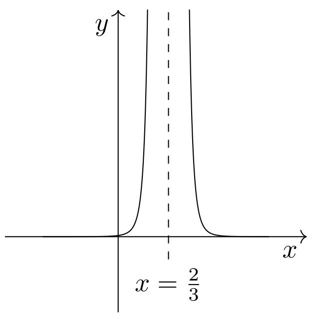
So, $(2, -0.5)$ is further from the parabola.

3134. (a) One sector of the polygon is a triangle with sides (R, R, l), where l is the side length. The angle at the centre is $\frac{360°}{n}$. Splitting into two right-angled triangles, the side length l is then given by $2R \sin \frac{180°}{n}$. Multiplying by n gives the perimeter result.

(b) The above result for the perimeter tells us that $n \sin \frac{180°}{n} = \frac{P}{2R}$. In the limit as $n \to \infty$, the n-gon approaches a circle, with radius R and perimeter $2\pi R$. Therefore,

$$\lim_{n\to\infty} n \sin \tfrac{180°}{n} = \frac{2\pi R}{2R} = \pi, \text{ as required.}$$

3135. This is an input transformation of $y = \frac{1}{x^6}$, which is a reciprocal graph of even degree. $y = \frac{1}{x^6}$ has a sextuple (akin to double) asymptote at $x = 0$. Replacing x by $3x - 2$ translates this by $2\mathbf{i}$, then stretches it by scale factor $\frac{1}{3}$ in the x direction. This takes the asymptote to $x = \frac{2}{3}$:

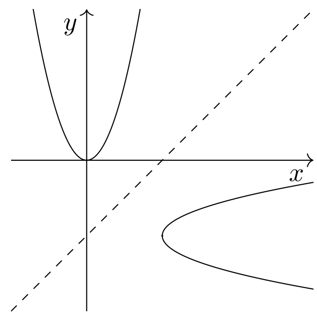


3136. The denominator is a quadratic in x^2, which has $\Delta = -19 < 0$. So, the denominator is never zero, meaning that the function is well defined over $\mathbb{R}$.

3137. The trivial case is where every pupil in the group got exactly the same scores X and Y, in which case there is no variation at all, and correlation is undefined.

Otherwise, with M set as a constant, the equation $M = aX + bY$ can be considered as a straight line on a set of (X, Y) axes. This line has $-$ve gradient, since $a, b > 0$. All of the pupils in the group have scores which lie on this line, which means there is perfect negative correlation between their scores.

3138. The image is as shown:



The vertex has been transformed from $(0, 0)$ to $(4, -4)$. So, we translate the parabola $x = y^2$ by $4\mathbf{i} - 4\mathbf{j}$, which gives $x - 4 = (y + 4)^2$.

3139. The width of the domain is 1, so the average value is given by the definite integral between 0 and 1. Hence, the equation is

$$\int_0^1 x^2 - a\,dx = \int_0^1 x^3\,dx$$
$$\Longrightarrow \left[\tfrac{1}{3}x^3 - ax\right]_0^1 = \left[\tfrac{1}{4}x^4\right]_0^1$$
$$\Longrightarrow \tfrac{1}{3} - a = \tfrac{1}{4}$$
$$\Longrightarrow a = \tfrac{1}{12}.$$

3140. (a) $0^4 = 0^4 - 0^2$, so $(0, 0) \in S$.

(b) Let x be such that $0 < |x| < 1$. Then $x^4 < x^2$, so $x^4 - x^2 < 0$. Hence, since $y^4 \geq 0$, there are no (x, y) points in S, apart from the origin, for which $|x| < 1$.

3141. Using $B(6, p)$, the relevant probabilities are

$$\mathbb{P}(X = 0) = (1 - p)^6 = q^6,$$
$$\mathbb{P}(X = 0, 1, 2) = q^6 + 6q^5(1 - q) + 15q^4(1 - q)^2.$$

Substituting into $\mathbb{P}(X = 0 \mid X \leq 2) = \frac{4}{31}$,

$$\frac{q^6}{q^6 + 6q^5(1 - q) + 15q^4(1 - q)^2} = \frac{4}{31}$$
$$\Longrightarrow 9q^6 - 96q^5 + 60q^4 = 0$$
$$\Longrightarrow q = 0, 2/3, 10.$$

We reject $q = 0$, for which the original probability is undefined, and $q = 10 > 1$. So, $p = 1/3$.

3142. Substituting for y,

$$x^3 + 2(4 - x)^3 = 24$$
$$\Longrightarrow x^3 - 24x^2 + 96x - 104 = 0.$$

Using a polynomial solver, $x = 2, 11 \pm \sqrt{69}$. Since $x, y \in \mathbb{Z}$, the solution is $x = 2$, $y = 2$.

3143. Since $\mathrm{f}(x)$ is a solution, $x\,\mathrm{f}'(x) + \mathrm{f}(x) = 1$. To sub $y = \mathrm{f}(x) + c$ into the DE, we find its derivative, which is the same as that of the original solution curve: $\frac{dy}{dx} = \mathrm{f}'(x)$. So, we require

$$\begin{aligned} x\,\mathrm{f}'(x) + \mathrm{f}(x) + c &= 1, \\ \Longrightarrow x\,\mathrm{f}'(x) + \mathrm{f}(x) &= 1 - c. \end{aligned}$$

But we know that $x\,\mathrm{f}'(x) + \mathrm{f}(x) = 1$, so $1 = 1 - c$. Hence, $c = 0$. So, the rogue mathematician is wrong: adding a constant c to one solution curve does not, in this case, produce another solution curve.

3144. (a) Let p be the probability that any carpenter in the population works left-handed. With this definition, the hypotheses are:

$$\begin{aligned} H_0 &: p = 0.1 \\ H_1 &: p < 0.1. \end{aligned}$$

(b) Given $X \sim B(n, 0.1)$, the probability of zero carpenters working left-handed is 0.9^n:

$$\begin{aligned} 0.9^n &= 0.01 \\ \Longrightarrow n &= \log_{0.9} 0.01 \approx 43.7. \end{aligned}$$

So, the critical region will be empty for $n \leq 43$, and will contain (at least) 0 for $n \geq 44$.

(c) For $n = 50$, $\mathbb{P}(X = 0) = 0.00515... < 1\%$ and $\mathbb{P}(X \leq 1) = 0.0337... > 1\%$. So, the c.r. is $\{0\}$. Assuming that H_0 is in fact true, the probability of rejection is 0.00515 (3sf).

3145. The third Pythagorean trig identity is

$$\cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta.$$

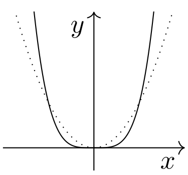
Setting $\theta = 3x$, this tells us that $\mathrm{h}(x)$ is constant, with value -1.

3146. For a point of inflection, the second derivative f'' must be zero *and must also change sign.*

A counterexample is $\mathrm{f}(x) = x^4$, with $\mathrm{f}''(x) = 12x^2$. This is zero at $x = 0$, but (as a square) doesn't change sign. The graph $y = x^4$ has a minimum at $x = 0$, not a point of inflection.

— Nota Bene —

The visual effect of the zero second derivative, in the case of $y = x^4$, is the particularly "snub-nosed" look to the graph, when compared to a parabola:



That the second derivative is zero points to the fact that the SP at $x = 0$ is *very* stationary. It doesn't say anything about direction, which is the relevant fact for a point of inflection.

3147. (a) Expanding, $(1 + x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + ...$ For $x = 0.08$, the terms in x^2 and above can be neglected, leaving $\sqrt{1.08} \approx 1.04$.

(b) Spotting that 1.08 is three times 0.36, which is a perfect square, $\sqrt{1.08} = \sqrt{3 \times 0.36} = \frac{6}{10}\sqrt{3}$. Substituting this into the result from (a),

$$\begin{aligned} \tfrac{6}{10}\sqrt{3} &\approx \tfrac{104}{100} \\ \Longrightarrow \sqrt{3} &\approx \tfrac{26}{15}. \end{aligned}$$

3148. We can write $\mathrm{f}(x) = (x - \alpha)^2\,\mathrm{g}(x)$, where $\mathrm{g}(x)$ is a polynomial. Differentiating by the product rule,

$$\begin{aligned} \mathrm{f}'(x) &= 2(x - \alpha)\,\mathrm{g}(x) + (x - \alpha)^2\,\mathrm{g}'(x) \\ &= (x - \alpha)\big(2\,\mathrm{g}(x) - (x - \alpha)\,\mathrm{g}'(x)\big). \end{aligned}$$

So, $\mathrm{f}'(x)$ has a factor of $(x - \alpha)$, as required.

3149. (a) Using the cosine rule,

$$\begin{aligned} \cos\theta &= \frac{n^2 + (n+1)^2 - (n+2)^2}{2n(n+1)} \\ &\equiv \frac{(n-3)(n+1)}{2n(n+1)} \\ &\equiv \frac{n-3}{2n}. \end{aligned}$$

The first Pythagorean trig identity gives

$$\begin{aligned} \sin\theta &= \sqrt{1 - \frac{(n-3)^2}{4n^2}} \\ &\equiv \sqrt{\frac{3(n^2 + 2n - 3)}{4n^2}} \\ &= \frac{\sqrt{3(n^2 + 2n - 3)}}{2n}. \end{aligned}$$

(b) Using $A_{\triangle} = \frac{1}{2}ab\sin C$,

$$\begin{aligned} \tfrac{1}{2}n(n+1)\frac{\sqrt{3(n^2 + 2n - 3)}}{2n} &= 84 \\ \Longrightarrow (n+1)\sqrt{3(n^2 + 2n - 3)} &= 336 \\ \Longrightarrow (n+1)^2(n^2 + 2n - 3) &= 37632. \end{aligned}$$

(c) We multiply out to get

$$n^4 + 4n^3 + 2n^2 - 4n - 37635 = 0.$$

Using a polynomial solver, and rejecting the negative root, $n = 13$.

3150. Making the trig ratio the subject of each, we then square the equations to give

$$\sec^2 t = \frac{(x-1)^2}{25}, \quad \tan^2 t = (y-3)^2.$$

Substituting these into the second Pythagorean trig identity, the Cartesian equation is

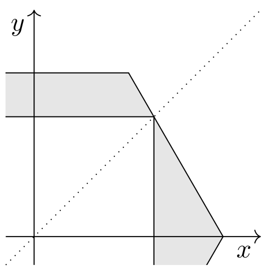
$$1 + (y-3)^2 = \frac{(x-1)^2}{25}.$$

3151. There are $4! = 24$ orders of $\{x_1, x_2, x_3, x_4\}$. The given fact $x_1 < x_4$ cuts the possibility space by half, as $x_1 < x_4$ and $x_4 < x_1$ are equally likely. This leaves 12 equally likely outcomes, of which one is successful. So, the probability is $\frac{1}{12}$.

3152. The first and third derivatives of $\sin x$ are $\cos x$ and $-\cos x$. So, the LHS is

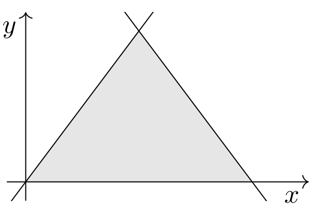
$$\begin{aligned}&\left(\frac{d^3y}{dx^3} + 2\frac{dy}{dx}\right)^2\\ &= (-\cos x + 2\cos x)^2\\ &\equiv \cos^2 x\\ &\equiv 1 - \sin^2 x\\ &= (1+y)(1-y), \text{ as required.}\end{aligned}$$

3153. Introducing (x, y) axes, the first quadrant is:



The equation of the sloped edge is $y = -\sqrt{3}x + \sqrt{3}$. We need the intersection of this with $y = x$, which is $x = y = \frac{1}{2}(3 - \sqrt{3})$. This gives the side length of the square as $3 - \sqrt{3}$, as required.

3154. The lines enclose a triangular region. The pairwise points of intersection are at $(0, 0)$, $(6, 0)$ and $(3, 4)$.



The base has length 6 and the perpendicular height is 4. This gives $A_\triangle = \frac{1}{2}bh = 12$, as required.

3155. Differentiating implicitly,

$$\begin{aligned}2x + 2y\frac{dy}{dx} &= 0\\ \implies \frac{dy}{dx} &= -\frac{x}{y}.\end{aligned}$$

Hence, the gradient of the normal is y/x, and the equation of the normal through (p, q) is

$$\begin{aligned}y - q &= \frac{q}{p}(x - p)\\ \implies y &= \frac{q}{p}x.\end{aligned}$$

This passes through the origin, as required.

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Although this question requires (in its wording) implicit differentiation, the result is more easily reached using circle geometry: $x^2 + y^2 = r^2$ is a circle centred on the origin, and the normals are radii.

3156. This is a quadratic in 3^{4x}. Using index laws,

$$\begin{aligned}&3^{8x+1} + 2 \times 9^{2x} = 1\\ \implies &3 \times (3^{4x})^2 + 2 \times 3^{4x} - 1 = 0\\ \implies &(3 \times 3^{4x} - 1)(3^{4x} + 1) = 0\\ \implies &3^{4x} = \tfrac{1}{3}, -1.\end{aligned}$$

We reject the latter root, as 3^{4x} is positive. This leaves $3^{4x} = \frac{1}{3}$, so $x = -\frac{1}{4}$.

3157. By the product rule, $\frac{dy}{dx} = 2xe^x + x^2e^x$. At $(2, 4e^2)$, this gives $m = 8e^2$. So, the equation of the tangent is $y - 4e^2 = 8e^2(x-2)$. Substituting $y = 0$,

$$\begin{aligned}-4e^2 &= 8e^2(x-2)\\ \implies -\tfrac{1}{2} &= x - 2\\ \implies x &= \tfrac{3}{2}, \text{ as required.}\end{aligned}$$

3158. The answer is yes in each case. In (a) and (c), you know the common ratio r with certainty. In (b), you don't. But you know r^2 with certainty, which tells you the value of all the odd-numbered terms.

(a) Yes,

(b) Yes,

(c) Yes.

3159. The discriminant of the quadratic is $\Delta = -36$. Since this is less than zero, the quadratic has no real roots. And, since the leading coefficient is positive, $f(x) > 0$ for all $x \in \mathbb{R}$. Hence, applying a modulus function does nothing, and $f(x) \equiv |f(x)|$, as required.

3160. There are two cases to check, depending on the sign of $(\sec\theta + \tan\theta)$. If this is positive, then the mod function is inactive:

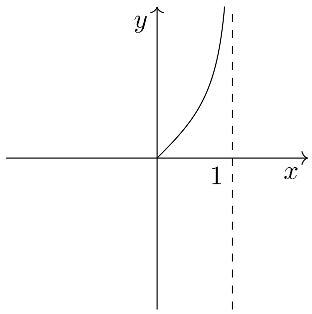
$$\begin{aligned}&\frac{d}{dx}\big(\ln(\sec\theta + \tan\theta) + c\big)\\ \equiv\ &\frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta}\\ \equiv\ &\frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta}\\ \equiv\ &\sec\theta.\end{aligned}$$

With a minus sign inside the logarithm, we cancel a factor of $(-\sec\theta - \tan\theta)$ instead, yielding the same result. Combining these two as an integral,

$$\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + c, \text{ as required.}$$

3161. This is false. The implication goes one way but not the other. Consider $f(x) = (x - 0.5)^2$. This has a (double) root at $x = 0.5$, but shows no sign change between $x = 0$ and $x = 1$.

3162. (a) The ranges, for $t \in [0, \pi/2)$, are $R_x = [0, 1)$ and $R_y = [0, \infty)$. We also know that, at $t = 0$, $\dot{x} = \dot{y} = 1$. So, the particle begins moving along $y = x$, before tending asymptotically to $x = 1$:



(b) The acceleration in the y direction is

$$\ddot{y} = 2\sec^2 t \tan t.$$

This tends to infinity as $t \to \frac{\pi}{2}$. By NII, the force would have to tend to infinity, which is not possible.

3163. If cubics $y = f_1(x)$ and $y = f_2(x)$ are tangent at $x = \alpha$, without them crossing, then the equation for intersections $f_1(x) - f_2(x) = 0$ has a double (and not triple) root at $x = \alpha$. So, $f_1(x) - f_2(x)$ has a factor of $(x - \alpha)^2$ (and not $(x - \alpha)^3$).

Since the leading coefficients of f_1 and f_2 differ, $f_1(x) - f_2(x)$ is cubic. Taking out the factor of $(x - \alpha)^2$ must leave a single factor $(x - \beta)$, where $\beta \neq \alpha$. The graphs cross at $x = \beta$. QED.

3164. The definite integral tells us that the mean value of $f(x)$, for $x \in [a, b]$, is 1. And, since f is linear, the mean value of $f(x)$ on $[a, b]$ occurs at the mean of a and b. Hence, the value of the function at $\frac{a+b}{2}$ is 1.

3165. Without loss of generality, we can use $Z \sim N(0, 1)$, and consider only $Z \geq 0$:

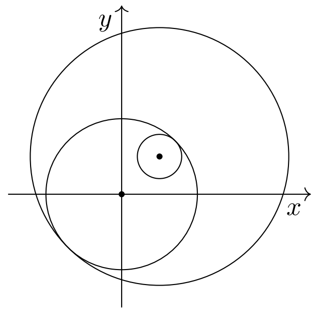
$$\begin{aligned}\mathbb{P}(Z > 2 \mid Z > 1) &= \frac{\mathbb{P}(Z > 2)}{\mathbb{P}(Z > 1)}\\ &= \frac{0.02275}{0.15866}\\ &= 0.143 \text{ (3sf)}.\end{aligned}$$

3166. Using a double-angle formula,

$$\begin{aligned}&\sin^2 x + \tfrac{\sqrt{3}}{2}\sin 2x = 0\\ \Longrightarrow\ &\sin^2 x + \sqrt{3}\sin x\cos x = 0\\ \Longrightarrow\ &\sin x(\sin x + \sqrt{3}\cos x) = 0\\ \Longrightarrow\ &\sin x = 0 \text{ or } \tan x = -\sqrt{3}.\end{aligned}$$

This gives $x = 0, \frac{2\pi}{3}, \pi, \frac{5\pi}{3}$.

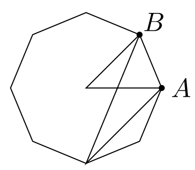
3167. The two boundary cases are as shown:



The distance between the centres is $\sqrt{2}/2$, so the radii are $1 \pm \sqrt{2}/2$. Hence,

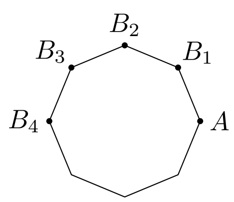
$$k \in \left(1 - \tfrac{\sqrt{2}}{2}, 1 + \tfrac{\sqrt{2}}{2}\right).$$

3168. (a) We can place vertices A and B without loss of generality:



The angle subtended at the centre by AB is 45°, so, by the angle at the centre theorem, the angle subtended at vertex C is 22.5°.

(b) There are four different location-types for B. The first three have probability $\frac{2}{7}$, and the last has probability $\frac{1}{7}$.



We use the method from (a), case by case. The probabilities p that $\angle ACB$ is acute are

$$\begin{aligned} B_1 &: p = 1 \\ B_2 &: p = \tfrac{5}{6} \\ B_3 &: p = \tfrac{4}{6} \\ B_4 &: p = 0. \end{aligned}$$

So, the overall probability that $\angle ACB$ is acute is $\frac{2}{7}\left(1 + \frac{5}{6} + \frac{4}{6}\right) = \frac{5}{7}$.

3169. Let the functions be

$$\mathrm{f}_1(x) = \frac{\mathrm{g}_1(x)}{\mathrm{h}_1(x)} \text{ and } \mathrm{f}_2(x) = \frac{\mathrm{g}_2(x)}{\mathrm{h}_2(x)},$$

where $\mathrm{g}_1, \mathrm{g}_2, \mathrm{h}_1, \mathrm{h}_2$ are polynomial functions. We set up a composition as

$$\mathrm{f}_1\mathrm{f}_2(x) = \frac{\mathrm{g}_1\left(\frac{\mathrm{g}_2(x)}{\mathrm{h}_2(x)}\right)}{\mathrm{h}_1\left(\frac{\mathrm{g}_2(x)}{\mathrm{h}_2(x)}\right)}.$$

When expanded, the numerator and denominator of the main fraction both consist of terms which are rational functions of x. Let the polynomial $\mathrm{P}(x)$ be the product of all of the denominators of these terms.

If we multiply the numerator and denominator of the main fraction by $\mathrm{P}(x)$, then we eliminate all of the inlaid fractions. The fraction is now a quotient of polynomials, so $\mathrm{f}_1\mathrm{f}_2$ is a rational function. □

3170. (a) Differentiating the position,

$$\mathbf{v} = \begin{pmatrix} 2e^t \\ -2e^{2t} \\ 2e^t \end{pmatrix} \text{ ms}^{-1}.$$

(b) Using 3D Pythagoras,

$$\begin{aligned} &\left(2e^t\right)^2 + \left(-2e^{2t}\right)^2 + \left(2e^t\right)^2 = 8 \\ \Longrightarrow\ & e^{4t} + 2e^{2t} - 2 = 0 \\ \Longrightarrow\ & e^{2t} = \frac{-2 \pm \sqrt{12}}{2} = -1 \pm \sqrt{3}. \end{aligned}$$

We reject the negative root as $e^{2t} > 0$, which gives $e^{2t} = \sqrt{3} - 1$, so $t = \frac{1}{2}\ln(\sqrt{3} - 1)$ s.

3171. Let the triangle have sides of length $(a, a, 2h)$. From the perimeter, $2a + 2h = 50$, so $a + h = 25$. Splitting the triangle in two,

$$\begin{aligned} & h\sqrt{a^2 - h^2} = 120 \\ \Longrightarrow\ & h^2\left(a^2 - h^2\right) = 14400. \end{aligned}$$

Substituting $a = 25 - h$,

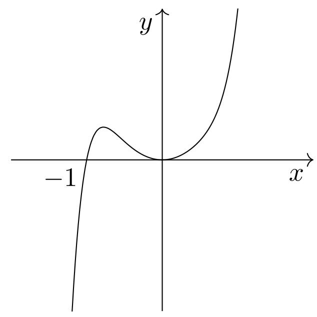
$$\begin{aligned} & h^2\left((25 - h)^2 - h^2\right) = 14400 \\ \Longrightarrow\ & 2h^3 - 25h^2 + 576 = 0. \end{aligned}$$

Using a polynomial solver, the integer root is $h = 8$. This gives a $(17, 17, 16)$ triangle.

3172. Solving for roots, $x^2(x^5 + 1) = 0$. So, either $x = 0$ (double root) or $x^5 = -1$, which gives $x = -1$. This is a single root, as seen in the factorisation

$$x^5 + 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1).$$

The quartic has no real roots. Overall, we have a positive curve of odd degree, with a single root at $x = -1$ and a double root (point of tangency) at the origin:



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The fact that $x = -1$ is a single root can also be seen in the fact that $y = x^5$ crosses the line $y = -1$ with a non-zero gradient.

3173. (a) The line $y = \sin\theta$ intersects the unit semicircle at $(\pm\cos\theta, \sin\theta)$. So, the area subtended by the central arc is $\pi - 2\theta$ radians. The area of the segment is therefore

$$\begin{aligned} A_{\text{seg}} &= A_{\text{sec}} - A_{\triangle} \\ &= \tfrac{1}{2}(\pi - 2\theta) - \tfrac{1}{2}\sin(\pi - 2\theta) \\ &= \tfrac{1}{2}(\pi - 2\theta) - \tfrac{1}{2}\sin 2\theta. \end{aligned}$$

The area of the semicircle is $\frac{1}{2}\pi$, so

$$\begin{aligned} & \tfrac{1}{2}(\pi - 2\theta) - \tfrac{1}{2}\sin 2\theta = \tfrac{1}{4}\pi \\ \Longrightarrow\ & 4\theta + 2\sin 2\theta = \pi. \end{aligned}$$

(b) We set up the iteration $\theta_{n+1} = \frac{1}{4}(\pi - 2\sin 2\theta_n)$. Running this starting at $\theta_0 = 0$, the iteration converges: $\theta \to 0.415856 = 0.4159$ (4sf).

(c) Defining $\mathrm{f}(\theta) = 4\theta + 2\sin 2\theta - \pi$,

$$\begin{aligned} \mathrm{f}(0.41585) &= -0.000037... < 0 \\ \mathrm{f}(0.41595) &= 0.000631... > 0. \end{aligned}$$

So, to 4sf, $\theta = 0.4159$ radians.

3174. A generic straight line is $y = mx + c$. This has derivative m, so we require $m + mx + c \equiv 2x + 1$. Equating coefficients, $m = 2$ and $c = -1$. So, the straight line is $y = 2x - 1$.

3175. (a) $\mathbb{P}(\text{all odd}) = \frac{5}{10} \times \frac{4}{9} \times \frac{3}{8} = \frac{1}{12}$.

(b) There are 8 options for the lowest number: $0, 1, ..., 7$. So, there are 8 sets of the form $\{n, n+1, n+2\}$:

$$\mathbb{P}(\{n, n+1, n+2\}) = \frac{8}{{}^{10}\mathrm{C}_3} = \tfrac{1}{15}.$$

3176. By symmetry, the distance between the rollers must remain constant. This is half the length of the block. While the rollers are either side of the centre, the block will not tip. Hence, tipping will occur when the rollers are at the centre and the leftmost edge.

At this point, the block will have moved a distance $l/4$ relative to the rollers, which will, by symmetry, have moved a distance $l/4$ relative to the ground. So, the displacement in question is $l/2$.

3177. Let $u = 5x - 1$, so that $dx = \frac{1}{5}du$. We also know that $x = \frac{1}{5}(u+1)$. The u limits are $u = 4$ to $u = 49$. Enacting the substitution,

$$\begin{aligned}
&\int_1^{10} 21x^2\sqrt{5x-1}\, dx \\
= &\int_4^{49} 21 \cdot \tfrac{1}{25}(u+1)^2\sqrt{u} \cdot \tfrac{1}{5}du.
\end{aligned}$$

Expanding the brackets and simplifying, this is

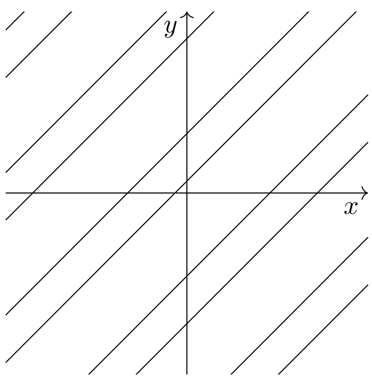
$$\begin{aligned}
&\tfrac{21}{125}\int_4^{49} u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\, du \\
&= \tfrac{21}{125}\left[\tfrac{2}{7}u^{\frac{7}{2}} + \tfrac{4}{5}u^{\frac{5}{2}} + \tfrac{2}{3}u^{\frac{3}{2}}\right]_4^{49} \\
&= \tfrac{21}{125}\left(\tfrac{3734584}{15} - \tfrac{7088}{105}\right) \\
&= 41816, \text{ as required.}
\end{aligned}$$

3178. Setting $y = \pm 2$,

$$\begin{aligned}
&\left(\pm 4 - \sqrt{3}x\right)^2 = 4 - x^2 \\
\implies &12 \mp 8\sqrt{3}x + 4x^2 = 0.
\end{aligned}$$

The discriminant is $\Delta = 0$. So, in each $\pm$ case, there is a double root. This signifies that each of the lines $y = \pm 2$ is tangent to the curve.

3179. The boundary equation is $\sin(x+y) = 1/2$, which gives $x+y = \pi/6, 5\pi/6$ and additions of $2n\pi$ to these. The solution set consists of parallel lines:



Since $5\pi/6 - \pi/6 = 2\pi/3$, the distances between the lines are in the ratio $2 : 1$. The broader regions are red, and the thinner regions are blue. So, the probability of picking a red point is $2/3$.

3180. At $2°$, we have $18 = (40\cos 2°)t$ horizontally, giving $t = 0.45027...$ s. Vertically, this gives the height as

$$\begin{aligned}
h &= 1.5 + (40\sin 2°)t - 4.9t^2 \\
&= 1.14 \text{ (2dp)}.
\end{aligned}$$

Repeating the calculation at $3°$, the time of flight is $t = 0.45061...$ and the height is $h = 1.448...$. So, to the nearest cm, $h \in [1.14, 1.45]$ m.

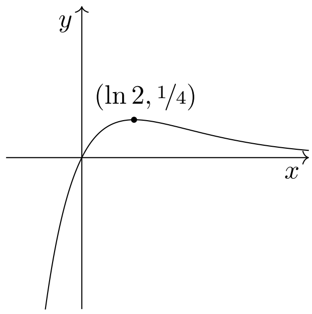
3181. Solving for roots,

$$\begin{aligned}
&e^{-x} - e^{-2x} = 0 \\
\implies &e^x - 1 = 0 \\
\implies &x = 0.
\end{aligned}$$

So, the curve passes through the origin. Next, we solve for SPs:

$$\begin{aligned}
&-e^{-x} + 2e^{-2x} = 0 \\
\implies &-e^x + 2 = 0 \\
\implies &x = \ln 2.
\end{aligned}$$

This gives a stationary point at $(\ln 2, 1/4)$. The behaviour for large x is: $x \to \infty$, $y \to 0$, and $x \to -\infty$, $y \to -\infty$. Putting all of the above together, the sketch is



3182. Solving for intersections,

$$\begin{aligned}
&4\sqrt{x} = 4x^2 + 7x \\
\implies &16x = 16x^4 + 56x^3 + 49x^2 \\
\implies &16x^4 + 56x^3 + 49x^2 - 16x = 0.
\end{aligned}$$

Using a polynomial solver, $x = 0, 1/4$. The area of the shaded region is given by

$$A = \int_0^{\frac{1}{4}} \sqrt{x} - \left(x^2 + \tfrac{7}{4}x\right) dx.$$

Using a definite integrator, $A = \frac{3}{128}$.

3183. The multiples of 17 form an arithmetic sequence $u_n = 17n$. Solving $17n = 1000$ gives $n = 58.8$, so there are 58 multiples of 17 in the first 1000 integers. These have sum

$$\tfrac{1}{2} \times 58 \times 59 \times 17 = 29087.$$

The sum of the first 1000 integers is

$$\tfrac{1}{2} \times 1000 \times 1001 = 500500.$$

Subtracting these, $500500 - 29087 = 471413$.

3184. The circle is $x^2 + y^2 = 1$. Substituting in, we have $\sin^2 t + \sin^2 2t = 1$. Using a double-angle formula,

$$\begin{aligned} & \sin^2 t + 4\sin^2 t \cos^2 t = 1 \\ \Longrightarrow\ & \sin^2 t + 4\sin^2 t(1 - \sin^2 t) = 1 \\ \Longrightarrow\ & 4\sin^4 t - 5\sin^2 t + 1 = 0 \\ \Longrightarrow\ & (4\sin^2 t - 1)(\sin^2 t - 1) = 0 \\ \Longrightarrow\ & \sin^2 t = \tfrac{1}{4}, 1 \\ \Longrightarrow\ & \sin t = \pm\tfrac{1}{2}, \pm 1. \end{aligned}$$

This gives six intersections: two at $(\pm 1, 0)$, and four at all combinations of signs in $(\pm 1/2, \pm\sqrt{3}/2)$.

3185. Differentiating by the chain rule,

$$\begin{aligned} & x \equiv e^{\ln x} \\ \Longrightarrow\ & 1 \equiv e^{\ln x}(\ln x)' \\ \Longrightarrow\ & (\ln x)' \equiv \frac{1}{e^{\ln x}} \\ & \equiv \frac{1}{x}, \text{ as required.} \end{aligned}$$

3186. The Cartesian equation of the first line (segment) is $x + y = 3$. The endpoints of the second line segment are $A : (-1, 17)$ and $B : (3, 1)$. Testing $x + y$ for A and B, we get 16 and 4. Since both values are greater than 3, A and B lie on the same side of $x + y = 3$. Hence, the line segments do not intersect.

3187. Using the product rule,

$$\frac{d}{dx}(xy) \equiv y + x\frac{dy}{dx}.$$

Using it again,

$$\begin{aligned} \frac{d^2}{dx^2}(xy) &\equiv \frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2} \\ &\equiv 2\frac{dy}{dx} + x\frac{d^2y}{dx^2}, \text{ as required.} \end{aligned}$$

3188. The line is the perpendicular bisector of $(0, 0)$ and (p, q). So, it is the line with gradient $-p/q$ passing through $(p/2, q/2)$. The equation of this line is

$$\begin{aligned} & \frac{y - q/2}{x - p/2} = -\frac{p}{q} \\ \Longrightarrow\ & \frac{2y - q}{2x - p} + \frac{p}{q} = 0, \text{ as required.} \end{aligned}$$

3189. (a) This is false. A counterexample is $f(x) = x^2$ and $g(x) = x + 1$, with $\alpha = 0$. The function $fg(x) = (x + 1)^2$ is stationary at $x = -1$, but not at $x = 0$.

(b) This is true. By the chain rule, the derivative of $gf(x)$ is $g'(f(x))\,f'(x)$. This is zero at $x = \alpha$, because $f'(\alpha) = 0$.

3190. We divide top and bottom by a common factor, before taking the limit:

$$\begin{aligned} & \lim_{x\to e} \frac{1 - \ln x}{(\ln x)^2 - 1} \\ =\ & \lim_{x\to e} \frac{1 - \ln x}{(\ln x + 1)(\ln x - 1)} \\ =\ & \lim_{x\to e} \frac{-1}{\ln x + 1} \\ =\ & \frac{-1}{\ln e + 1} \\ =\ & -\tfrac{1}{2}, \text{ as required.} \end{aligned}$$

3191. The implication is only forwards. To get back from the derivatives to the original functions, one must integrate, which introduces arbitrary constants.

$$f(x) \equiv g(x) \implies f'(x) \equiv g'(x).$$

3192. The curves are reflections in the line $y = x$. So, their intersections lie on $y = x$. Solving for these,

$$\begin{aligned} & x + 1 = x^2 + x \\ \Longrightarrow\ & x = \pm 1. \end{aligned}$$

The area enclosed by the curves is then twice the area enclosed by $y + 1 = x^2 + x$ and $y = x$. This is

$$\begin{aligned} A &= 2\int_{-1}^{1} x - (x^2 + x - 1)\,dx \\ &= 2\Big[-\tfrac{1}{3}x^3 + x\Big]_{-1}^{1} \\ &= 2\Big(\big(\tfrac{2}{3}\big) - \big(-\tfrac{2}{3}\big)\Big) \\ &= \tfrac{8}{3}. \end{aligned}$$

3193. The function is a positive sextic. So, it must have a global minimum. Completing the square,

$$x \longmapsto \left(x^3 + \tfrac{1}{2}\right)^2 + \tfrac{3}{4}.$$

So, the range over $\mathbb{R}$ is $\{y \in \mathbb{R} : y \geq 3/4\}$.

3194. Subtracting the first equation from the second, $x - 2y = 2$; subtracting the first from the third, $2x - 4y = 4$. These equations are multiples of one another, so any (x, y) point on the line $x = 2y + 2$ satisfies both of them. Substituting this into the original equations,

$$\begin{aligned}
(2y+2) + y + z = 1 &\iff 3y + z = -1,\\
2(2y+2) - y + z = 3 &\iff 3y + z = -1\\
3(2y+2) - 3y + z = 5 &\iff 3y + z = -1.
\end{aligned}$$

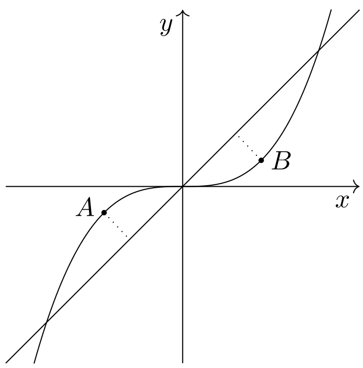
Hence, for any $y \in \mathbb{R}$, we can set $x = 2y + 2$ and $z = -1 - 3y$ and get a distinct point (x, y, z). There are infinitely many $y \in \mathbb{R}$, proving the result.

3195. The population is large, so each chosen individual has the same (negligibly different) probability of being above the 90th percentile. The distribution for each is $X \sim B(4, 0.1)$. This gives

$$\begin{aligned}
&\mathbb{P}(X = 2, 3, 4)\\
&= {}^4\mathrm{C}_2 \tfrac{9^2}{10^4} + {}^4\mathrm{C}_3 \tfrac{9}{10^4} + {}^4\mathrm{C}_4 \tfrac{1}{10^4}\\
&= 0.0523.
\end{aligned}$$

3196. Assume, for a contradiction, that a^2b^3 is a square. Then, for some integer k, $a^2b^3 = k^2$. Consider the factors of b on each side. Both a^2 and k^2 must have even numbers of prime factors of b, while b^3 has exactly three. So, there is an odd number of factors of b on the LHS, and an even number of factors of b on the RHS. This is a contradiction. Hence, the product is not a square. □

3197. Graphically, fixed points of f are intersections of $y = \mathrm{f}(x)$ and $y = x$. We are told that there are three such intersections. Consider, therefore, the points A, B between these intersections for which the distance from $y = \mathrm{f}(x)$ to $y = x$ is locally maximised.

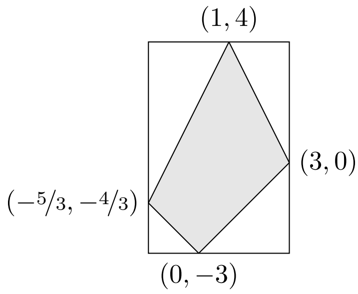


Since the dotted lines shown are perpendicular to both $y = \mathrm{f}(x)$ and $y = x$, the gradient of the curve is $\mathrm{f}'(x) = 1$ at these points. QED.

3198. Rearranging to $x = \frac{1}{2}(1 - u)$, we use the binomial expansion:

$$\begin{aligned}
&-8x^3 + 16x^2 - 10x - 2\\
&= -(1-u)^3 + 4(1-u)^2 - 5(1-u) - 2\\
&\equiv u^3 + u^2 - 4.
\end{aligned}$$

3199. The vertices of the quadrilateral are at $(1, 4)$, $(3, 0)$, $(0, -3)$ and $(-5/3, -4/3)$. So, the minimum bounding rectangle of the quadrilateral has area $14/3 \times 7 = 98/3$.



The areas of the unshaded triangles are, clockwise from top-right, 4, $9/2$, $25/18$ and $64/9$. So, the area of the shaded quadrilateral is

$$\tfrac{98}{3} - \left(4 + \tfrac{9}{2} + \tfrac{25}{18} + \tfrac{64}{9}\right) = \tfrac{47}{3}, \text{ as required.}$$

3200. The angle α between a line $y = mx + c$ and the x axis satisfies $\tan\alpha = m$. So, rearranging to $\tan\alpha = \cos x_0$, we know that $\mathrm{f}'(x_0) = \cos x_0$. This holds for any x_0, so $\mathrm{f}'(x) = \cos x$. Integrating, the function f must be of the form $\mathrm{f}(x) = \sin x + c$.

—— End of 32nd Hundred ——